Problem 1.14

Suppose f is a function of two variables (y and z) only. Show that the gradient $\nabla f = (\partial f/\partial y)\hat{\mathbf{y}} + (\partial f/\partial z)\hat{\mathbf{z}}$ transforms as a vector under rotations, Eq. 1.29. [*Hint:* $(\partial f/\partial \bar{y}) = (\partial f/\partial y)(\partial y/\partial \bar{y}) + (\partial f/\partial z)(\partial z/\partial \bar{y})$, and the analogous formula for $\partial f/\partial \bar{z}$. We know that $\bar{y} = y\cos\phi + z\sin\phi$ and $\bar{z} = -y\sin\phi + z\cos\phi$; "solve" these equations for y and z (as functions of $\bar{y}$ and $\bar{z}$), and compute the needed derivatives $\partial y/\partial \bar{y}$, $\partial z/\partial \bar{y}$, etc.]

Solution

Suppose there's a yz-coordinate system, and the axes are rotated counterclockwise by an angle ϕ in order to make a new $\bar{y}\bar{z}$-coordinate system. A point represented by (y, z) has the new coordinates $(\bar{y}, \bar{z})$ given by

$$\bar{y} = y\cos\phi + z\sin\phi$$

$$\bar{z} = -y\sin\phi + z\cos\phi.$$

Solve this system of two equations for y and z.

$$y = \bar{y}\cos\phi - \bar{z}\sin\phi$$

$$z = \bar{y}\sin\phi + \bar{z}\cos\phi$$

The gradient of a scalar function f in the $\bar{y}\bar{z}$-coordinate system is

$$\overline{\nabla} f = \left\langle \frac{\partial f}{\partial \bar{y}}, \frac{\partial f}{\partial \bar{z}} \right\rangle.$$

Use the chain rule to determine its components in terms of the old variables.

$$\frac{\partial f}{\partial \bar{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{y}} = \frac{\partial f}{\partial y}(\cos\phi) + \frac{\partial f}{\partial z}(\sin\phi)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \bar{z}} = \frac{\partial f}{\partial y}(-\sin\phi) + \frac{\partial f}{\partial z}(\cos\phi)$$

These equations can be represented in matrix form as follows (Eq. 1.29).

$$\begin{pmatrix} \frac{\partial f}{\partial \bar{y}} \\ \frac{\partial f}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \tag{1.29}$$

Therefore, the gradient transforms under rotations as a vector.